AD-A096 401 MASSACHUSETTS UNIV AMHERST DEPT OF MATHEMATICS AND S--ETC F/6 12/1
A CHARACTERIZATION OF THE WARING DISTRIBUTION.(U)
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(6) A CHARACTERIZATION OF THE WARING DISTRIBUTION.

By

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(9) AFOSR Technical Report No. 9

(14) TR-9

(16) 2304 (17) A5

(11) January 1981

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[1]Research sponsored by the ~~Air Force Office~~ of Scientific Research, AFSC, USAF, under Grant AFOSR-80-0219 (15)

A CHARACTERIZATION OF THE WARING DISTRIBUTION.

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SUMMARY. In this note the Waring distribution is characterized by the following property: For a positive integer-valued random variable X, $P(x=r) = p_r$, $r=1,2,\ldots$, and with a finite mean μ define two new random variables Y and Z by

$$P(Y=r) = q_r = \left(\sum_{k=r+1}^{\infty} p_k + ap_r\right)/(\mu+a)\ ,\ r=0,1,2,\ldots,$$

$$P(Z=r) = q_r' = (r+b)p_r/(\mu+b)\ ,\ r=1,2,\ldots,$$

where $a\geq 0$ and b are constants with $b-a+1>0$. Then Z and Y truncated at 0 have the same distribution if and only if X has a Waring distribution.

Let X be a positive integer-valued random variable (r.v.) with

$$P(X=r) = p_r,\ r=1,2,\ldots, \tag{1}$$

Let X have a finite mean μ. One can define two new classes of r.v!s Y and Z by

$$P(Y=r) = q_r = \left(\sum_{k=r+1}^{\infty} p_k+ap_r\right)/(\mu+a)\ ,\ r=0,1,\ldots, \tag{2}$$

Research sponsored by the Air Force Office of Scientific Research, ESC, USAF, under Grant AFOSR-80-0219.

$$P(Z=r) = q_r = (\mu+b)p_r/(\mu+b) , \ r=1,2,..., \tag{3}$$

where $a \geq 0$ and $b>-1$ are constants. Distributions of the type (2) have been defined and studied in the literature. See Johnson and Kotz (1969,p.261). It is natural to ask: For what distributions X,Y and Z are essentially the same r.v's? Since Y takes the values 0,1,..., while Z the values 1,2,..., it is necessary to truncate Y at 0 , and then it turns out that the above property is a unique property of the Waring distribution. For our purpose we define the Waring distribution of Irwin (1963) by

$$P(W=r) = (\lambda-c)c^{[r-1]}/\lambda^{[r]} , \ r=1,2,..., \tag{4}$$

where

$$\lambda-c>1 , \ c>0;$$

and

$$c^{[r]} = c(c+1) \ldots (c+r-1) , \ r=1,2,..., \ c^{[0]}=1.$$

We are now ready to characterize the Waring distribution(4):

Theorem: Let X be a positive integer-valued random variable given by(1) with a finite mean μ . Define Y and Z by (2) and (3) respectively for some constants $a \geq 0$, b satisfying $b-a+1>0$. Then Y truncated at 0 has the same distribution as Z if and only if X has the Waring distribution(4).

Proof : "Only if" part: Let q_r and q'_r be given by (2) and (3) for some $a \geq 0$ and $b-a+1>0$ with $q'_r = (1-q_0)^{-1} q_r$, $r=1,2,\ldots,$

Then substituting for the p's in (2) from (3) we have

$$q_r = \{(\mu+b)/(\mu+a)\}\{\sum_{k=r+1}^{\infty} q'_k/(k+b) + a q'_r/(r+b)\}, \quad r=1,2,\ldots, \qquad (5)$$

$$= (\mu+a)^{-1}, \quad r=0,$$

From (5) we obtain

$$q_r - q_{r+1} = \left[q'_{r+1}/(r+1+b) + a\{q'_r/(r+b) - q'_{r+1}/(r+1+b)\}\right](\mu+b)/(\mu+a)$$

$$r=1,2,\ldots,$$

from which we get the recurring relation

$$q'_{r+1} = \{(r+b-ax)(r+b+1)\} q'_r / \{(r+b+1+x-ax)(r+b)\}, \qquad (6)$$

where

$$x = (\mu+b)/\{(\mu+a)(1-q_0)\} = (\mu+b)/(\mu+a-1) \qquad (7)$$

Equation (6) yields

$$q'_{r+1} = \frac{(r+b-ax)\ldots(1+b-ax)(r+1+b)}{(r+1+b+x-ax)\ldots(2+b+x-ax)(1+b)} q'_1 \qquad (8)$$

$$r=1,2,\ldots$$

However, from (5) and the assumed relation $q'_r = (1-q_0)^{-1} q_1$ it follows that

4

$$q_1' = q_0\,(1+b)/\{(1-q_0)(1+b+x-ax)\} \qquad (9)$$

Finally from (3), (5)-(9) we obtain

$$p_r = (1+b-ax)^{[r-1]}/(1+b+x-ax)^{[r]}\ ,\quad r=1,2,\ldots \qquad (10)$$

which is Waring (4) with

$$c = (1+b-ax) \text{ and } \lambda=(1+b+x-ax).$$

It is easily seen from (5)-(7) and the assumption $b-a+1>0$ that $c = (\mu-1)(b-a+1)/(\mu+a-1)>0$, and since $x>1$ we also have $\lambda-c=x>1$.

"If" part: Let the p's be given by (4). Then using the Waring expansion $(\lambda-c)^{-1} = \sum_{r=0}^{\infty} c^{[r]}/\lambda^{[r+1]}$, which is valid for $\lambda>c>0$, we obtain

$$\sum_{k=r+1}^{\infty} p_k = (c+r-1)p_r/(\lambda-c)$$

from which and the fact that $\mu=c/(\lambda-c-1)+1$ one could easily verify that $q_r' = (1-q_0)^{-1}q_r$ where q_r, q_r' are given by (2) and (3) with $a = (b-c+1)/(\lambda-c)$, b arbitrary but $b-c+1>0$. Since $\lambda-c$ is assumed to be greater than 1, we have for the above a and b that $a<b-c+1$ or $0<c<b-a+1$. This completes the proof of the theorem.

Note that our theorem covers the characterization of the Yule distribution: In the "Only if" part the Yule distribution corresponds to the case $a=b=0$ and in the "if" part to $c=1$.

There is, in fact, a certain connection to the characterization of the Yule distribution given by Krishnaji (1970). Krishnaji showed that X has the Yule distribution

$$P(X=r) = (d-1)r!/\{(d+1)(d+2)\ldots(d+r)\}, \quad r=1,2,\ldots \tag{11}$$

where $d-1>0$ if and only if X and the greatest integer in UX trucated at 0 have the same distribution. Here U is a uniform r.v. on (0,1) independently distributed of X. Now if in the "Only if" part of our theorem we take a=b=0, (5) will reduce to

$$q_r = \sum_{k=r+1}^{\infty} q'_k/k, \quad r=0,1,\ldots$$

which is $P([UZ]=r)$ as has been shown by Krishnaji. Here U is uniform on (0,1) distributed independently of Z (and hence of X) and $[x]$ denotes the greatest integer in x. Thus Krishnaji's result as applied to Z and $[UZ]$ yields that Z has the Yule distribution (11), and X in turn has the Yule distribution (4) with c=1. Of course, our "Only if" part gives the same result.

REFERENCES

Irwin, J.O. (1963). Inverse factorial series as frequency distributions. Proceedings of the International Symposium On Discrete Distributions, Montreal, 159-74.

Johnson, N.L. and Kotz, S. (1969). Discrete Distributions, Wiley, [illegible]

Krishnaji, N. (1970). A Characteristic Property of the Yule distribution. Sankhya A, 32, 343-46.

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SECURITY CLASSIFICATION OF THIS PAGE (WHEN DATE ENTERED)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER AFOSR-TR-81-0266	2. GOVT ACCESSION NO. AD-A096 401	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) A CHARACTERIZATION OF THE WARING DISTRIBUTION		5. TYPE OF REPORT & PERIOD COVERED INTERIM
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) RAMESH M. KORWAR		8. CONTRACT OR GRANT NUMBER(s) AFOSR-80-0219
9. PERFORMING ORGANIZATION NAME AND ADDRESS The University of Massachusetts Department of Mathematics & Statistics Amherst, MA 01003		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS 61102F 2304/A5
11. CONTROLLING OFFICE NAME AND ADDRESS Air Force Office of Scientific Research Bolling Air Force Base, Washington DC 20332		12. REPORT DATE January 1981
		13. NUMBER OF PAGES 6
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release: distribution unlimited

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENT TARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identity by block number)

Characterization
Discrete distribution
Truncated distribution
Waring distribution
Yule distribution

20. ABSTRACT (Continue on reverse side if necessary and identity by block number)

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